Factor supply elasticity of factor price given a Cobb-Douglas production[1]

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February 2019

We assume the following Cobb-Douglas production function:

$$y = x_1^{\alpha} x_2^{1-\alpha} \qquad (0 \leq \alpha \leq 1) \tag{1}$$

where y is total output and x_1, x_2 are the inputs of production factors 1,2 (e.g. labour and capital). The coefficient α represents the constant output elasticity of production factor 1, and so does (1- α) with respect to production factor 2. As α and (1- α) add up to unity, the production function in equation (1) implies constant returns to scale.

Assuming supply of production factor 1 to be inelastic with respect to its price, but subject to autonomous changes, and furthermore assuming supply of production factor 2 to be fixed, minimization of production costs implies equality of price and marginal productivity with respect to production factor 1:

$$p_1 = \frac{\partial y}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{1-\alpha} = \frac{\alpha y}{x_1} \tag{2}$$

Note that the endowment of production factor 1 equals a fraction α of output y:

$$x_1 p_1 = \alpha y \tag{2a}$$

A similar equation with respect to production factor 2 can be derived analogously:

$$x_2 p_2 = (1-\alpha) y \tag{2b}$$

So, the endowment of production factor 2 equals a fraction (1- α) of output y, meaning that the shares of both production factors in total production cost add up to total revenue from output.

The factor supply elasticity of factor price $\varepsilon_{\mathrm{ii}}$ is defined as follows:

$$\varepsilon_{\mathrm{ii}} \equiv \frac{\partial(\log p_{\mathrm{i}})}{\partial(\log x_{\mathrm{i}})} \tag{3}$$

[1] An example of the practical relevance of the present exercise is the assessment of the macroeconomic gains from immigration accruing to the resident population: the so-called *immigration surplus*. See for instance Borjas, G.J., 1999. The economic analysis of immigration. In: Ashenfelter, O., Card, D. (Eds.), Handbook of Labor Economics, vol. 3. North-Holland, Amsterdam, pp. 1697 – 1760.

Where p_{i} is the price of production factor i. The numerator can be rewritten for small changes as follows:

$$\partial(\log p_{\mathrm{i}}) \approx \frac{1}{p_{\mathrm{i}}} \cdot \partial p_{\mathrm{i}} \tag{4}$$

The denominator can be rewritten for small changes as follows:

$$\partial(\log x_{\mathrm{i}}) \approx \frac{1}{x_{\mathrm{i}}} \cdot \partial x_{\mathrm{i}} \tag{5}$$

After some substitution and rearrangement we get the following result with respect production factor 1:

$$\varepsilon_{11} \approx \frac{\partial p_1}{\partial x_1} \cdot \frac{x_1}{p_1} = \frac{x_1}{p_1} \cdot \alpha(\alpha - 1)x_1^{\alpha-2}x_2^{1-\alpha} = \frac{x_1}{p_1} \cdot (\alpha - 1)\alpha y x_1^{-2} \tag{4}$$

Using equation (2a) this result comes down to:

$$\varepsilon_{11} \approx \alpha - 1 \tag{5}$$

Analogously, we can derive for production factor 2:

$$\varepsilon_{22} \approx -\alpha \tag{6}$$

So, given a Cobb-Douglas production function, the factor supply elasticity of factor price is constant and equals the corresponding output elasticity minus one.